Eur. Phys. J. C 4, 723–739 (1998)
DOI 10.1007/s100529800776

THE EUROPEAN
PHYSICAL JOURNAL C

Supersymmetric structures in 4-D Yang–Mills theory

Jussi Kalkkinen, Antti J. Niemi[a]

Department of Theoretical Physics, Uppsala University, P.O. Box 803, S-75108, Uppsala, Sweden
(e-mail: jussi.kalkkinen@teorfys.uu.se, antti.niemi@teorfys.uu.se)

Received: 18 July 1997 / Revised version: 15 August 1997 / Published online: 23 February 1998

Abstract. Recently there has been much progress in understanding confinement in the N=2 supersymmetric Yang–Mills theory. Here we shall investigate how these results could be extended to explain color confinement in the ordinary Yang–Mills theory. In particular, we inquire whether confinement in the N=2 theory can be related to color confinement in the ordinary Yang–Mills theory in the framework of Parisi–Sourlas dimensional reduction. For this we study the partition function of the ordinary Yang–Mills theory in different regimes. Our analysis reveals that an intimate connection indeed exists between these two approaches.

1 Introduction

The N=2 supersymmetric Yang–Mills theory with its supersymmetry broken to N=1, materializes [1] the qualitative picture of confinement developed in [2]. However, despite remarkable progress we still lack an explanation why color and quarks confine in the standard QCD. In the present paper we shall address the problem of color confinement in ordinary Yang–Mills theory. In particular, we wish to understand how the results obtained in the N=2 supersymmetric case could be extended. There are some indications that a direct connection might exist:

Several years ago it has been proposed [3,4] that color confinement in the ordinary Yang–Mills theory is a consequence of the Parisi–Sourlas dimensional reduction [5]. In this approach one argues that in the infrared limit the Yang–Mills ground state can be approximated by a medium of randomly distributed white noise (Gaussian) color-electric and color-magnetic fields. An effective field theory description of this medium entails the Parisi–Sourlas supersymmetric Yang–Mills theory, and confinement follows from the ensuing D=4 $\rightarrow$ D=2 dimensional reduction.

Subsequently it has been observed [6] that the Parisi–Sourlas supersymmetric Yang–Mills theory is also related to the four-dimensional topological Yang–Mills theory [7]. Since the topological Yang–Mills theory is a twisted version of the N=2 supersymmetric theory, it becomes natural to expect that confinement in the N=2 theory should also admit a Parisi–Sourlas interpretation. In particular, there should be an intimate relationship between the confinement mechanisms in the N=2 supersymmetric and the ordinary Yang–Mills theories.

Here we shall establish that a direct connection between the N=2 and the Parisi–Sourlas approaches to confinement indeed exists. For this we employ the background field method in path integral formalism [8]. We explicitly perform the summation over all classical Yang–Mills background field configurations, and obtain an effective field theory which exhibits the Parisi–Sourlas supersymmetry. On the other hand when we restrict the summation to either self-dual or anti-self-dual background field configurations, we find an effective field theory that contains the N=2 supersymmetric Yang–Mills theory in a similar manner. Furthermore, by analyzing the N=2 supersymmetry algebra that emerges in our construction we conclude that in our case the N=2 theory is necessarily restricted to points of its moduli space where massless dyons appear. According to [1] these massless dyons should condense which leads to confinement in the N=2 theory. Consequently our results imply that the dyon condensate is described by the Parisi–Sourlas supersymmetric Yang–Mills theory, as a medium of randomly distributed color-electric and color-magnetic fields.

In the next section we first shortly review the background field formalism. We then describe how we use this formalism to implement a summation either over all possible background fields, or over either self-dual or anti-self-dual background fields. In Sect. 3 we consider the effective field theory formulation that emerges when we sum over all possible background fields. We show that the result contains the Parisi–Sourlas supersymmetric extension of ordinary Yang–Mills theory. In Sect. 4 we proceed towards the self-dual approximation. For this we interpret the effective measure that appears in our path integral in terms of topological Yang–Mills theory. We then apply this result to show how our self-dual approximation entails topological Yang–Mills theory instead of the Parisi–Sourlas one. In Sect. 5 we (un)twist the action of Lorentz transformations

[a] Supported by Göran Gustafsson Foundation for Science and Medicine and by NFR Grant F-AA/FU 06821-308

in our topological theory, and show that it corresponds to the minimal N=2 supersymmetric Yang–Mills theory at a point in its moduli space where massless dyons appear. In Sect. 6 we compare our approaches. In the appendices we describe some technical aspects of our work.

2 Background field quantization

In the present paper we shall argue that both Parisi–Sourlas and N=2 supersymmetric theories emerge when an ordinary Yang–Mills theory is formulated in an appropriate background. For this we consider the background field quantization [8, 9] of a SU(N) Yang–Mills theory with a gauge field C_μ^a and a field strength

$$F_{\mu\nu} \;=\; [D_\mu, D_\nu] \;=\; \partial_\mu C_\nu - \partial_\nu C_\mu + [C_\mu, C_\nu]\,. \tag{1}$$

We normalize the Lie algebra generators T^a so that the classical action is

$$S_{\rm YM}(C) \;=\; \int \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \;\equiv\; \int \frac{1}{4} F^2_{\mu\nu}\,. \tag{2}$$

We consider the Euclidean partition function

$$Z \;=\; \int_{\mathcal{A}/\mathcal{G}} [\,\mathrm{d}C]\exp\{-S_{\rm YM}(C)\} \tag{3}$$

in the background field formalism. Here the functional integration is extended over the space of gauge equivalence classes $\mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ is the affine space of connections on the pertinent principal bundle and $\mathcal{G}$ are the gauge transformations. The integration can be performed over $\mathcal{A}$ provided we fix the gauge symmetry e.g. using the standard Faddeev–Popov procedure. The usual nilpotent BRST operator is

$$\Omega \;=\; \mathbf{c}\; D_\mu(C) H_\mu \;+\; \frac{1}{2}\mathbf{b}[\mathbf{c},\mathbf{c}] \;+\; t\bar{\mathbf{c}}\,. \tag{4}$$

The fields $\mathbf{c}, \mathbf{b}$ are the ghost and its momentum, $\bar{\mathbf{c}}, \bar{\mathbf{b}}$ the anti-ghost and its momentum and t the Lagrange-multiplier. The operator Ω acts through the (formal) Poisson-brackets given e.g. for the gauge field C and its momentum H by the relation

$$\{H^a_\mu, C^b_\nu\} = -\delta^{ab}_{\mu\nu}\,. \tag{5}$$

We fix the gauge by using the gauge-fermion

$$\Psi = \bar{\mathbf{b}}\Big(t + D_\mu(A) C_\mu\Big)\,, \tag{6}$$

where A_μ is a background field that satisfies the Lorentz gauge condition $\partial \cdot A = 0$.

2.1 The background gas approximation

The partition function (3) becomes in the background formalism

$$\begin{aligned} Z[A,J] &= \int_{\mathcal{A}} [\,\mathrm{d}C][\,\mathrm{d}\bar{\mathbf{b}}][\,\mathrm{d}\mathbf{c}][\,\mathrm{d}t]\exp\{-S_{\rm YM}(C) - \{\Omega,\Psi\}\} \\ &= \int_{\mathcal{A}} [\,\mathrm{d}C]\;\delta\Big(D(A)\cdot C\Big)\det\|D(A)\cdot D(C)\| \\ &\qquad \exp\{-S[C]\}\,. \end{aligned} \tag{7}$$

Since the functional integral is performed over an affine space $\mathcal{A}$ we can shift the integration variable: For this we represent the gauge field C_μ as a linear combination

$$C_\mu \;=\; A_\mu \;+\; Q_\mu\,. \tag{8}$$

The functional integral now becomes

$$\begin{aligned} Z[A,J] = \int_{\mathcal{A}} [\,\mathrm{d}C]\;\delta\Big(D(A)\cdot Q\Big)\;\det\|D(A)\cdot D(A+Q)\| \\ \exp\Big(-S[A+Q] - J\cdot Q\Big)\,, \end{aligned} \tag{9}$$

where we have coupled a current J_μ to the fluctuation field Q_μ and used $\partial\cdot A = 0$. The way this coupling is done turns particularly important in what follows.

The partition function Z is invariant under local gauge transformations if A_μ transforms as a gauge field and if J_μ transforms in the adjoint representation of the gauge group. This, together with the assumption that A_μ satisfy the Lorentz gauge condition, means that Z is a locally defined functional on the space of gauge orbits $\mathcal{A}/\mathcal{G}$.

In what follows, we shall find a *globally* on $\mathcal{A}/\mathcal{G}$ defined expression for the partition function. In the above, A_μ was understood as a (classical) background field configuration around which the (possibly not small) quantum fluctuations Q_μ take place. In particular, if the base manifold has no boundary, or Q has compact support, then the (integrated) second Chern character

$${\rm Ch}_2(F_A) \;= -\frac{1}{16\pi^2}\;\int F_{\mu\nu}\tilde{F}_{\mu\nu}(A) \tag{10}$$

of the quantum field remains constant in the background gauge. This is due to the fact that then the second Chern characters can differ only by an exact differential of a gauge-invariant 3-form

$$\begin{aligned} &{\rm Ch}_2(F_{A+Q}) = {\rm Ch}_2(F_A) \\ &\quad - \frac{1}{8\pi^2}\,\mathrm{d}\;{\rm tr}\Big(Q\wedge(D_A Q + \frac{2}{3}Q\wedge Q + 2F_A)\Big)\,. \end{aligned} \tag{11}$$

In order to guarantee that our partition function also describes nonperturbative phenomena correctly, we should in some consistent fashion extend the partition function out of the neighborhood of the chosen connection A. For this, it is useful to notice following [9] that if we choose the current $J = J(A)$ so that

$$\frac{\delta Z[J,A]}{\delta J} = 0\,, \tag{12}$$

we then find

$$Z[J(A),A] = \mathrm{e}^{-\Gamma[A]}\,, \tag{13}$$

where $\Gamma[A]$ is the standard Yang–Mills effective action. We can now turn to the problem into a *classical* construction, and study the classical partition function

$$Z[J] = \sum_{\mathcal{F}(A)=0} Z[J,A]\,\mathrm{e}^{2\pi\mathrm{i}\theta(A)k}\,, \tag{14}$$

where the sum is taken over all those fields $A \in \mathcal{A}/\mathcal{G}$ that satisfy the yet unspecified condition $\mathcal{F}(A) = 0$. The summand is merely the Boltzmann weight appended with an A-dependent "chemical potential" for the instanton number $k = \mathrm{Ch}_2(A)$.

The expression (14) is easiest analyzed in the path integral formalism. There is, however, no tractable way to deal with the absolute values of a fluctuation determinant that will appear in this treatment. It is therefore necessary to make a simplifying assumption about the A-dependence of the potential $\theta(A)$. For this, let λ_n be the eigenvalues of the fluctuation matrix,

$$\frac{\delta\mathcal{F}}{\delta A}\psi_n \;=\; \lambda_n \psi_n \; . \tag{15}$$

We define

$$\sum_{\lambda_n<0} 1 = \frac{1}{2}\sum_{\lambda_n} 1 \;-\; \frac{1}{2}\sum_{\lambda_n} \mathrm{sign}(\lambda_n) = \frac{1}{2}\zeta_{\mathcal{F}} \;-\; \frac{1}{2}\eta_{\mathcal{F}} \; , \tag{16}$$

where $\zeta_{\mathcal{F}}$ is the ζ-function of the fluctuation matrix

$$\zeta_{\mathcal{F}}(s) \;=\; \sum_{\lambda_n} |\lambda_n|^{-s} \; , \tag{17}$$

and $\eta_{\mathcal{F}}$ is its η-invariant,

$$\eta_{\mathcal{F}}(s) \;=\; \sum_{\lambda_n} \mathrm{sign}(\lambda_n)|\lambda_n|^{-s} \; , \tag{18}$$

both evaluated at $s \to 0$. Now the sign of the fluctuation determinant is given by

$$\mathrm{sign}\det||\frac{\delta\mathcal{F}}{\delta A}|| = \exp\Big(\mathrm{i}\pi\sum_{\lambda_n<0} 1\Big) = \exp\Big(\frac{\mathrm{i}\pi}{2}(\zeta_{\mathcal{F}}-\eta_{\mathcal{F}})\Big) \; . \tag{19}$$

We shall conveniently choose the form of the potential, namely, let

$$\theta[A] = \theta_0 + \frac{1}{4k}\Big(\zeta_{\mathcal{F}}(A) - \eta_{\mathcal{F}}(A)\Big) \; , \tag{20}$$

which is formally supposed to be valid even for $k = 0$. The reason for this is that now we can write the summation over the background fields A as an integral

$$Z[J] = \int_{\mathcal{A}/\mathcal{G}} [\,\mathrm{d}A]\; \delta\Big(\mathcal{F}(A)\Big)\; \det\|\frac{\delta\mathcal{F}(A)}{\delta A}\|\; Z[A,J] \times\; \exp(-2\pi\mathrm{i}\theta_0 k) \; . \tag{21}$$

The role of the phase factors is simply to cancel the absolute value signs that a delta-functional would produce in the path-integral treatment. This yields the natural extension to the case where the moduli space $\mathcal{M} = \{A \in \mathcal{A}/\mathcal{G} \mid \mathcal{F}(A) = 0\}$ is not discrete but a manifold with singularities. The continuous case necessitates, however, a proper treatment of zero modes as in [7], an issue which we shall not treat in detail here. Note that we also leave the global normalization of $Z[J]$ unspecified.

Let us next turn to the question of gauge fixing. In the path integral

$$\begin{aligned} Z[J] = \int_{\mathcal{A}} [\,\mathrm{d}A]\; &\delta\Big(\mathcal{F}(A)\Big)\; \det\|\frac{\delta\mathcal{F}(A)}{\delta A}\| \\ \times\; &\delta\Big(\partial\cdot A\Big)\; \det\|\partial\cdot D(A)\| \\ \times\; &\int_{\Omega^1(M)} [\,\mathrm{d}Q]\; \delta\Big(D(A)\cdot Q\Big) \\ \times\; &\det\|D(A)\cdot D(A+Q)\| \\ \times\; &\exp\Big(-S[A+Q] - 2\pi\mathrm{i}k\theta_0 - J\cdot Q\Big) \; , \end{aligned} \tag{22}$$

there are no unfixed gauge symmetries. The expression can be reinterpreted to originate from the path integral quantization of the classical theory $S_{\mathrm{YM}}[A+Q]$ that possesses the following two local infinitesimal gauge symmetries:

1. The gauge symmetry of the background field A is given by the infinitesimal transformations

$$\begin{aligned} \delta A_\mu &= D_\mu(A)\;\varepsilon \\ \delta Q_\mu &= [Q_\mu, \varepsilon] \; . \end{aligned} \tag{23}$$

The generating BRST operator and the gauge fermion used above are

$$\begin{aligned} \varOmega_1 &= \mathbf{c}\Big(D_\mu(A)E_\mu \;+ [Q_\mu, P_\mu]\Big) + \;\frac{1}{2}\mathbf{b}[\mathbf{c},\mathbf{c}] \;+\; t\bar{\mathbf{c}} \\ \varPsi_1 &= \bar{\mathbf{b}}\Big(t + \partial_\mu A_\mu\Big) \; . \end{aligned} \tag{24}$$

2. The gauge symmetry of the fluctuation field Q is given by

$$\begin{aligned} \delta A_\mu &= 0 \\ \delta Q_\mu &= D_\mu(A+Q)\;\varepsilon \; . \end{aligned} \tag{25}$$

The BRST operator and the gauge fermion are

$$\begin{aligned} \varOmega_2 &= \mathbf{u}D_\mu(A+Q)P_\mu \;+\; \frac{1}{2}\mathbf{v}[\mathbf{u},\mathbf{u}] \;+\; h\bar{\mathbf{u}} \\ \varPsi_2 &= \bar{\mathbf{v}}\Big(h + D_\mu(A)Q_\mu\Big) \; . \end{aligned} \tag{26}$$

Here we used Poisson-brackets

$$\{E^a_\mu, A^b_\nu\} = -\delta^{ab}_{\mu\nu} \; , \quad \{P^a_\mu, Q^b_\nu\} = -\delta^{ab}_{\mu\nu} \; . \tag{27}$$

These nilpotent BRST operators do not commute mutually, but they can be combined into a nilpotent operator by adding a ghost term

$$\varOmega = \varOmega_1 + \varOmega_2 + [\mathbf{u},\mathbf{c}]\mathbf{v} \; . \tag{28}$$

The resulting BRST operator has a simple interpretation: Let us make a change of variables

$$\begin{aligned} A_+ &= A + Q \\ A_- &= A \end{aligned} \tag{29}$$

and accordingly for momenta, so that the transformation has a trivial Jacobian and preserves the Poisson brackets. This, and a similar transformation in the ghost sector, brings the BRST operator into the form

$$\Omega = \Omega_- + \Omega_+ \; , \tag{30}$$

where the now mutually commuting BRST operators are

$$\Omega_\pm = \mathbf{u}_\pm D_\mu(A_\pm)E_{\pm,\mu} \; + \; \frac{1}{2}\mathbf{v}_\pm[\mathbf{u}_\pm, \mathbf{u}_\pm] + h_\pm \bar{\mathbf{u}}_\pm \tag{31}$$

in obvious notation.

The partition function can thus be written in a geometric form

$$\begin{aligned} Z[J] = \int [\,\mathrm{d}A_+][\,\mathrm{d}\mathbf{u}_+][\,\mathrm{d}\bar{\mathbf{v}}_+][\,\mathrm{d}t_+][\,\mathrm{d}A_-][\,\mathrm{d}\mathbf{u}_-][\,\mathrm{d}\bar{\mathbf{v}}_-][\,\mathrm{d}t_-] \\ \times\; \delta\Big(\mathcal{F}(A_-)\Big)\det\|\frac{\delta\mathcal{F}(A_-)}{\delta A_-}\| \\ \times\; \exp\Big(- S[A_+] - 2\pi \mathrm{i}\theta_0 \mathrm{Ch}_2(F_{A_-}) \\ -\{\Omega, \Psi_- + \Psi_+\} - J\cdot(A_+ - A_-)\Big) \; . \end{aligned} \tag{32}$$

This expression is a well defined integral over a globally defined density over the connection space $\mathcal{A}_+ \oplus \mathcal{A}_-$, and no more refers to a particular neighborhood of some preferred connection. In addition the functional $Z[J]$ is still invariant under the diagonal part of the gauge transformations $\mathcal{G}_+ \oplus \mathcal{G}_-$, as J transforms as an adjoint field.

Now we can use the freedom to choose the gauge at will: in particular, we can factorize the path integral into the form

$$\begin{aligned} Z[J] = \int_{\mathcal{A}/\mathcal{G}} [\,\mathrm{d}A_-]\exp(-S_{\mathcal{F}}[A_-]) \\ \times \int_{\mathcal{A}/\mathcal{G}} [\,\mathrm{d}A_+]\exp(-S_{\mathrm{YM}}[A_+]) \\ \times\; \exp(J\cdot(A_+ - A_-)) \; . \end{aligned} \tag{33}$$

In the following we shall study two different cases

$$\mathcal{F}(A) = \begin{cases} F_A^+ & \text{instanton background} \\ D_\mu(A)F_{A,\mu\nu} & \text{classical background.} \end{cases} \tag{34}$$

For both choices of $\mathcal{F}$ we shall find – modulo regularization issues that we shall deal with later – that in the case of vanishing current $J = 0$ the path integral factorizes completely into a product of the Euler characteristic of the space of gauge orbits $\mathcal{X}(\mathcal{A}/\mathcal{G})$ and the original Yang–Mills partition function. The coupling to the external current J_μ plays a crucial rôle here: it determines what the physically relevant coupling to the fields in the theory is. Since we choose to quantize fluctuations around classical configurations, this is where we should couple the external field as well. Other currents probe other aspects of the same quantum field theory.

It is interesting to note that integrating the partition function $Z[J]$ against a Gaussian weight in the current J would yield (perturbatively) a mass term for the fluctuation field Q. We shall later see that in the classical backround -case the effective action $S_{\mathcal{F}} = S_{\mathrm{TYM}}$ (cf. Sect. 4) possesses a translation symmetry in the gauge field: this would imply that in that case the current J really couples to Ω_{TYM}-cohomology classes.

The maybe awkward choice of the chemical potential can be a posteriori justified by the observations that we indeed get a globally defined expression on $\mathcal{A}/\mathcal{G}$, which does not depend on the details of the choice of the background condition $\mathcal{F} = 0$.

2.2 A Morse-theoretic approach

The fact that we could above consider either all classical backgrounds or merely instantons, can be explained using Morse theoretic arguments. In this section we shall rederive the previous result formally from a different starting point.

For this we represent as above the gauge field C_μ as a linear combination

$$C_\mu \;=\; A_\mu \;+\; Q_\mu \; , \tag{35}$$

and select A_μ to satisfy the classical equation of motion,

$$D_\mu F_{\mu\nu}(A) \;=\; \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] \;=\; 0 \; . \tag{36}$$

The field Q_μ describes perturbative quantum fluctuations around A_μ, and the classical field A_μ satisfies nontrivial boundary conditions. The Chern character of A_μ coincides with the second Chern character of C_μ as we saw previously.

If we define

$$G_{\mu\nu} \;=\; D_\mu Q_\nu - D_\nu Q_\mu \; , \tag{37}$$

where the covariant derivative is w.r.t. the classical field A_μ, we can write the action (2) as

$$\begin{aligned} -S_{\mathrm{YM}}(A+Q) = -\frac{1}{4}F_{\mu\nu}^2 \;-\; Q_\mu D_\nu F_{\mu\nu} \\ -\frac{1}{2}F_{\mu\nu}[Q_\mu, Q_\nu] - \;\frac{1}{4}G_{\mu\nu}^2 \\ -\;\frac{1}{2}G_{\mu\nu}[Q_\mu, Q_\nu] \;-\frac{1}{4}\;[Q_\mu, Q_\nu]^2 \; . \end{aligned} \tag{38}$$

Since A_μ solves the Yang–Mills equation (36), the Q-linear term $Q_\mu D_\nu F_{\mu\nu}$ in (38) actually vanishes. However, eventually we shall promote A_μ to an off-shell field so that it will cease to be constrained by (36). In that case the Q-linear term does not vanish, and in anticipation of this we have included it also here.

In terms of these background variables the path integral (3) becomes

$$\begin{aligned} Z_{\mathrm{YM}} = \sum_{A_\mu} \int [\,\mathrm{d}Q]\exp\Big\{ \int -\frac{1}{4}F_{\mu\nu}^2 \;-\; Q_\mu D_\nu F_{\mu\nu} \\ -\;\frac{1}{2}F_{\mu\nu}[Q_\mu, Q_\nu] - \;\frac{1}{4}G_{\mu\nu}^2 \;-\; \frac{1}{2}G_{\mu\nu}[Q_\mu, Q_\nu] \\ -\;\frac{1}{4}\;[Q_\mu, Q_\nu]^2 \;\Big\} \; . \end{aligned} \tag{39}$$

(In the following we shall not write the various gauge fixing terms, or the coupling to the external current explicitly.) The summation extends over all solutions A_μ of the Yang–Mills equation of motion, and the integral over Q_μ is subject to trivial boundary conditions as was explained above. Since the moduli space of the classical field A_μ is generically nontrivial (e.g. for a k-instanton in a SU(2)-theory it is a $(8k-3)$-dimensional manifold), the summation over A_μ should actually be viewed as an integration over the relevant moduli. Here we represent it as

$$\sum_{A_\mu} = \int [\,\mathrm{d}A]\delta(D_\mu F_{\mu\nu}) \left| \det || \frac{\delta D_\mu F_{\mu\nu}}{\delta A_\rho} || \right| . \tag{40}$$

Using the ζ-function techniques for $\mathcal{F}(A) = D\cdot F_A$ that were introduced above, we can rewrite the path integral (39) as

$$\begin{aligned} Z_{\mathrm{YM}} = \int [\,\mathrm{d}A][\,\mathrm{d}Q]\delta(DF)\det||\frac{\delta DF}{\delta A}|| \\ \times \exp\Big\{ \int -\frac{1}{4}F_{\mu\nu}^2 \; - \; Q_\mu D_\nu F_{\mu\nu} \; - \; \frac{1}{2}F_{\mu\nu}[Q_\mu, Q_\nu] \\ - \frac{1}{4}G_{\mu\nu}^2 \; - \; \frac{1}{2}G_{\mu\nu}[Q_\mu, Q_\nu] \; - \; \frac{1}{4}[Q_\mu, Q_\nu]^2 \\ - \frac{\mathrm{i}}{2}\pi(\zeta_{\mathrm{YM}} - \eta_{\mathrm{YM}})(A) \Big\} . \end{aligned} \tag{41}$$

Eventually we shall promote the classical background field A_μ into a dynamical off-shell field. In that case both $\zeta_{\mathrm{YM}}(A)$ and $\eta_{\mathrm{YM}}(A)$ become highly nontrivial functionals of A_μ and their exact evaluation becomes prohibitively complicated. Luckily, for the present purposes this appears unnecessary: We shall be mostly interested in the infrared limit of the Yang–Mills theory, and unless we attempt to compute higher order corrections, besides the renormalization of the coupling constant the contribution from these phase factors becomes irrelevant in this limit. Consequently it is sufficient to proceed with

$$\begin{aligned} Z_{\mathrm{YM}} \approx \int [\,\mathrm{d}A][\,\mathrm{d}Q]\delta(D_\mu F_{\mu\nu})\det||\frac{\delta D_\mu F_{\mu\nu}}{\delta A_\rho}|| \\ \times \exp\Big\{ \int -\frac{1}{4}F_{\mu\nu}^2 \; - \; Q_\mu D_\nu F_{\mu\nu} - \; \frac{1}{2}F_{\mu\nu}[Q_\mu, Q_\nu] \\ - \frac{1}{4}G_{\mu\nu}^2 \; - \; \frac{1}{2}G_{\mu\nu}[Q_\mu, Q_\nu] \\ - \frac{1}{4}\;[Q_\mu, Q_\nu]^2 \Big\} . \end{aligned} \tag{42}$$

In the next section we shall establish that the path integral (42) entails the Parisi–Sourlas supersymmetric Yang–Mills theory. In the subsequent sections we then explain how the topological Yang–Mills theory emerges when we consider an approximation of (42), obtained by summing over instantons.

The measure in (42) can be interpreted as the Poincaré–Hopf representation (see Appendix A) of the Euler characteristic $\mathcal{X}(\mathcal{A}/\mathcal{G})$ of the space of the gauge equivalence classes $\mathcal{A}/\mathcal{G}$,

$$\mathcal{X}(\mathcal{A}/\mathcal{G}) = \int [\,\mathrm{d}A]\delta(DF)\det||\frac{\delta DF}{\delta A}||$$

$$= \sum_{DF=0} \mathrm{sign}\det||\frac{\delta DF}{\delta A}|| . \tag{43}$$

Motivated by the fact that the Euler characteristic is at least in finite dimensions independent of the Morse function [10], we introduce alternative representations of (43). In infinite dimensions the result might depend on the regulator. Of particular interest are the representations obtained with the first-order (anti-)self-duality equations

$$F^{\pm} \; = \; F_{\mu\nu} \; \pm \; \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \; = \; 0 , \tag{44}$$

where the + refers to anti-self-dual configurations and − to self-dual ones. For finite action configurations these equations specify the instantons, and the pertinent representations of the Euler characteristic (43) are

$$\mathcal{X}(\mathcal{A}/\mathcal{G}) \; = \; \int [\,\mathrm{d}A] \; \delta(F^{\pm})\det||\frac{\delta F^{\pm}}{\delta A}|| . \tag{45}$$

We then proceed to approximate the Yang–Mills path integral by instead of (42) using

$$\begin{aligned} Z_{\mathrm{YM}} \approx \sum_{F^{\pm}=0} \mathrm{sign}\det||\frac{\delta F^{\pm}}{\delta A}|| \\ \times \; \int [\,\mathrm{d}Q]\exp\Big\{ \int -\frac{1}{4}F_{\mu\nu}^2(A+Q) \Big\} \\ \approx \; \int [\,\mathrm{d}A][\,\mathrm{d}Q]\delta(F^{\pm})\det||\frac{\delta F^{\pm}}{\delta A}|| \\ \times \; \exp\Big\{ \int -\frac{1}{4}F_{\mu\nu}^2 \; - \; Q_\mu D_\nu F_{\mu\nu} \\ - \; \frac{1}{2}F_{\mu\nu}[Q_\mu, Q_\nu] - \; \frac{1}{4}G_{\mu\nu}^2 \; - \; \frac{1}{2}G_{\mu\nu}[Q_\mu, Q_\nu] \\ - \frac{1}{4}\;[Q_\mu, Q_\nu]^2 \Big\} . \end{aligned} \tag{46}$$

We must again take the instanton zero modes into account by using collective coordinates and by inserting a suitable operator into the path integral to cancel the ghost number anomaly.

Notice that unlike in the conventional, semiclassical instanton approximation, in (46) we sum either over the self-dual or over the anti-self-dual configurations. Since we are only interested in the partition function, this is sufficient. However, in general we expect that a summation which extends only over self-dual or over anti-self-dual configurations violates cluster decomposition. For this reason, it might be more appropriate to rewrite (46) so that we sum both over self-dual and anti-self-dual configurations. This means that we represent the Euler characteristic by

$$\begin{aligned} \mathcal{X}(\mathcal{A}/\mathcal{G}) = \frac{1}{2}\Big(\sum_{F^+=0} \mathrm{sign}\det||\frac{\delta F^+}{\delta A}|| \\ + \; \sum_{F^-=0} \mathrm{sign}\det||\frac{\delta F^-}{\delta A}|| \Big) . \end{aligned} \tag{47}$$

But due to the obvious symmetry between the self-dual and anti-self-dual configurations, in the following it will be sufficient to consider only one of these contributions. When necessary, we can always introduce an explicit symmetrization.

3 Parisi–Sourlas supersymmetry

We shall first analyze the summation over all classical solutions,

$$Z_{\rm YM} \approx \int [\,{\rm d}Q][\,{\rm d}A]\delta(DF)\det||\frac{\delta DF}{\delta A}|| \times \exp\{-\int \frac{1}{4}F^2_{\mu\nu}(A+Q)\}\,. \tag{48}$$

We wish to show that (48) can be interpreted in terms of the Parisi–Sourlas supersymmetric Yang–Mills theory.

Using the general formalism that we have described in Appendix B, we introduce one commuting variable π_μ and two anticommuting variables ψ_μ and $\bar{\mathcal{P}}_\mu$ and write (48) as

$$\begin{aligned} Z_{\rm YM} \approx \int & [\,{\rm d}Q][\,{\rm d}A][\,{\rm d}\pi][\,{\rm d}\psi][\,{\rm d}\bar{\mathcal{P}}] \\ & \times \exp\Big\{ \int -\frac{1}{4}F^2_{\mu\nu}(A+Q) \\ & + \pi_\nu D_\mu F_{\mu\nu} \;-\; \bar{\mathcal{P}}_\mu[F_{\mu\nu},\psi_\nu] \\ & - \frac{1}{2}(D_\mu\psi_\nu - D_\nu\psi_\mu)(D_\mu\bar{\mathcal{P}}_\nu - D_\nu\bar{\mathcal{P}}_\mu) \Big\}\,. \end{aligned} \tag{49}$$

Here the first term depends on $A+Q$, while the remaining terms depend on A only.

We first argue that the linear combinations $A^+ \sim A+Q$ and $A^- \sim A$ determine two *independent* SU(N) gauge fields. For this we introduce a variable E^a_μ which is conjugate to the background field A^a_μ, and a variable P^a_μ which is conjugate to the fluctuation field Q^a_μ,

$$\{E^a_\mu, A^b_\nu\} \;=\; \{P^a_\mu, Q^b_\nu\} \;=\; -\,\delta^{ab}_{\mu\nu}\,, \tag{50}$$

and define the following linear combinations,

$$\begin{aligned} A^+_\mu &= A_\mu + Q_\mu \\ A^-_\mu &= A_\mu \\ E^+_\mu &= P_\mu \\ E^-_\mu &= E_\mu - P_\mu\,. \end{aligned} \tag{51}$$

Since the only nonvanishing Poisson brackets are

$$\{E^\pm_\mu, A^\pm_\nu\} \;=\; -\delta_{\mu\nu}\,, \tag{52}$$

we conclude that $A^+_\mu \sim A_\mu + Q_\mu$ and $A^-_\mu \sim A_\mu$ are indeed two *independent* canonical variables.

More generally, we extend these $\pm$ variables into a one-parameter family of $\pm$ variables by introducing a canonical conjugation generated by

$$\varPhi \;=\; -\,\tau E_\mu Q_\mu\,, \tag{53}$$

where τ is a parameter. This yields

$$\begin{aligned} A^+_\mu &\to e^{-\varPhi}A^+_\mu e^{\varPhi} &&= A_\mu + (1-\tau)Q_\mu \\ A^-_\mu &\to e^{-\varPhi}A^-_\mu e^{\varPhi} &&= A_\mu - \tau Q_\mu \\ E^+_\mu &\to e^{-\varPhi}E^+_\mu e^{\varPhi} &&= P_\mu + \tau E_\mu \\ E^-_\mu &\to e^{-\varPhi}E^-_\mu e^{\varPhi} &&= -\,P_\mu + (1-\tau)E_\mu\,. \end{aligned} \tag{54}$$

Since this is a canonical transformation, the Poisson brackets (52) are preserved. In particular, since both A^+_μ and A^-_μ gauge transform like an SU(N) gauge field

$$A^\pm_\mu \;=\; UA^\pm_\mu U^{-1} + U\partial_\mu U^{-1}\,, \tag{55}$$

we can indeed view them as independent gauge fields.

As a consequence the action in (50) separates into two independent terms, one that depends only on the gauge field A^+_μ and the other that depends only on the gauge field A^-_μ

$$S(A,Q,\pi,\psi,\bar{\mathcal{P}}) \;=\; S^+(A^+) \;+\; S^-(A^-;\pi,\psi,\bar{\mathcal{P}})\,, \tag{56}$$

where

$$S^+(A^+) = \int -\frac{1}{4}F^2_{\mu\nu}(A^+) \tag{57}$$

and

$$S^-(A^-;\pi,\psi,\bar{\mathcal{P}}) = \int \pi_\nu D_\mu F_{\mu\nu} \;-\; \bar{\mathcal{P}}_\mu[F_{\mu\nu},\psi_\nu] - \frac{1}{2}(D_\mu\psi_\nu - D_\nu\psi_\mu)(D_\mu\bar{\mathcal{P}}_\nu - D_\nu\bar{\mathcal{P}}_\mu)\,. \tag{58}$$

In particular, instead of the original SU(N) gauge symmetry we now have two *independent* SU(N) gauge symmetries acting on the fields $A^\pm$ respectively, i.e. we have a local $\mathrm{SU(N)}_+ \times \mathrm{SU(N)}_-$ gauge symmetry. However, since the fluctuation field

$$A^+_\mu \;-\; A^-_\mu \;=\; Q_\mu \tag{59}$$

obeys trivial boundary conditions the gauge fields $A^\pm_\mu$ are subject to the condition that their second Chern characters coincide,

$$\int F\tilde{F}(A^+) \;=\; \int F\tilde{F}(A^-)\,. \tag{60}$$

This ensures that *perturbatively* the path integral does factorize into independent $\pm$ partition functions: Since the Chern characters of $A^\pm_\mu$ coincide, their local i.e. perturbative fluctuations are independent. This is desirable, since we wish to reproduce the standard Yang–Mills perturbation theory. Only when nonperturbative effects become relevant, will there be a coupling between the $\pm$ sectors. This is due to the fact that we must eventually sum over all Chern characters, and after that the partition function naturally is no more factorizable.

We now proceed to relate the A^-_μ dependent part of our action to the Parisi–Sourlas supersymmetric Yang–Mills theory. For this we first observe that this action admits the following nilpotent BRST (Parisi–Sourlas) symmetry (see Appendix B)

$$\begin{aligned} \Omega A^-_\mu &= \psi_\mu \\ \Omega\psi_\mu &= 0 \\ \Omega\bar{\mathcal{P}}_\mu &= \pi_\mu \\ \Omega\pi_\mu &= 0\,, \end{aligned} \tag{61}$$

so that

$$\Omega \;=\; \psi_\mu E^-_\mu \;+\; \pi_\mu\bar\eta_\mu\,, \tag{62}$$

where we have introduced the conjugate variable

$$\{\bar{\mathcal{P}}_\mu,\bar\eta_\nu\} \;=\; -\delta_{\mu\nu}\,. \tag{63}$$

In particular, (58) can be represented as a BRST commutator,

$$S^-(A^-) \;=\; \{\Omega,\bar{\mathcal{P}}_\mu D_\nu F_{\mu\nu}\}\,. \tag{64}$$

If we define the space-time components of the Parisi–Sourlas supergauge field

$$\mathcal{A}_\mu \;=\; A^-_\mu + \theta\psi_\mu \;-\bar\theta\bar{\mathcal{P}}_\mu + \theta\bar\theta\pi_\mu\,, \tag{65}$$

we conclude from Appendix B that we can write (58) as

$$S^-(\mathcal{A}) \;=\; \frac{1}{4}\int \mathrm{d}x\,\mathrm{d}\bar\theta\,\mathrm{d}\theta\;\mathcal{F}^2_{\mu\nu}\,, \tag{66}$$

where $\mathcal{F}_{\mu\nu}$ denotes the space-time components of the Parisi–Sourlas field strength tensor,

$$\mathcal{F}_{\mu\nu} \;=\; \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu,\mathcal{A}_\nu]\,. \tag{67}$$

Notice that since Ω does not act on the A^+ field, the BRST symmetry (61) is actually an invariance of the entire action in (50). In particular, the path integral

$$Z_{\mathrm{YM}} \approx \int[\mathrm{d}Q][\mathrm{d}A][\mathrm{d}\pi][\mathrm{d}\psi][\mathrm{d}\bar{\mathcal{P}}] \times\exp\left\{\int -\frac{1}{4}F^2_{\mu\nu}(A^+) \;-\; \{\Omega,\Psi\}(\mathcal{A})\right\} \tag{68}$$

is invariant under local variations of the gauge fermion Ψ, and reproduces (50) when we select Ψ as in (64).

The action (66) suggests that we are indeed dealing with a Parisi–Sourlas supersymmetry. We now proceed to reveal the remaining θ and $\bar\theta$ structure. For this we first introduce the following more general gauge fermion

$$\Psi \;=\; \bar{\mathcal{P}}_\mu(D_\nu F_{\mu\nu} + \kappa\pi_\mu)\,, \tag{69}$$

where $\kappa \propto m^2$ is a mass scale. The ensuing path integral (68) is formally independent of κ, and when $\kappa\to 0$ we recover (64). The topological part of the action now becomes

$$\{\Omega,\Psi\} = \frac{1}{2}(D_\mu\psi_\nu - D_\nu\psi_\mu)(D_\mu\bar{\mathcal{P}}_\nu - D_\nu\bar{\mathcal{P}}_\mu) + \pi_\nu D_\mu F_{\mu\nu} \;+\; \bar{\mathcal{P}}_\mu[F_{\mu\nu},\psi_\nu] \;+\; \kappa\pi^2_\mu\,. \tag{70}$$

We can interpret this by introducing the full Parisi–Sourlas Yang–Mills field strength

$$\mathcal{F}_{\alpha\beta} \;=\; \partial_\alpha\mathcal{A}_\beta - (-)^{\alpha\beta}\partial_\beta\mathcal{A}_\alpha \;+\; [\mathcal{A}_\alpha,\mathcal{A}_\beta]\,, \tag{71}$$

where now $\alpha,\beta=\mu,\theta,\bar\theta$. If we define the full Yang–Mills action in the Parisi–Sourlas superspace

$$S_{\mathrm{PS}} \;=\; \int \mathrm{d}\theta\,\mathrm{d}\bar\theta\;\frac{1}{4}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\beta\alpha}\,, \tag{72}$$

and if we evaluate this action in the special case where

$$\mathcal{A}_\theta \;=\; \mathcal{A}_{\bar\theta} \;=\; 0\,, \tag{73}$$

we find (70) when we integrate over θ and $\bar\theta$. This suggests that the topological action (58) in our path integral (50) indeed determines a (4+2)-dimensional Parisi–Sourlas Yang–Mills theory.

In order to properly include the remaining $\mathcal{A}_\theta$ and $\mathcal{A}_{\bar\theta}$ components of the Parisi–Sourlas gauge field, we first introduce the following pairs of canonically conjugated variables

$$\{\mathcal{P}_\mu,\psi_\nu\} \;=\; \{\bar{\mathcal{P}}_\mu,\bar\psi_\nu\} \;=\; \{\pi_\mu,\lambda_\nu\} \;=\; -\delta_{\mu\nu}\,. \tag{74}$$

We then define

$$\mathcal{E}_\mu(y) \;=\; -\lambda_\mu + \theta\bar\psi_\mu + \mathcal{P}_\mu\bar\theta + \theta\bar\theta E^-_\mu\,, \tag{75}$$

so that we get the bracket

$$\{\mathcal{E}_\mu(y_1),\mathcal{A}_\nu(y_2)\} \;=\; -\delta_{\mu\nu}(y_1-y_2) \tag{76}$$

in the Parisi–Sourlas superspace. We then identify the nilpotent operator (62) as the BRST operator for the constraint

$$E^-_\mu \;\approx\; 0\,. \tag{77}$$

However, these constraints are not independent but are subject to the Gauss law

$$D_\mu E_\mu \;\approx\; 0\,, \tag{78}$$

which projects (62) to the space $\mathcal{A}/\mathcal{G}$. Such a linear relation among the constraints (77) means that the constraint algebra is reducible [11].

We now interpret Gauss law as a reducibility condition in the BRST operator (62). For this we introduce the following canonically conjugated Parisi–Sourlas superfields,

$$\begin{aligned} \mathcal{A}_\theta &= \eta - \theta\varphi - \bar\theta\lambda + \theta\bar\theta\bar\eta \\ \mathcal{E}_{\bar\theta} &= \bar{\mathcal{X}} + \theta\pi - \bar\theta p + \theta\bar\theta\mathcal{X} \end{aligned} \tag{79}$$

and

$$\begin{aligned} \mathcal{A}_{\bar\theta} &= \bar b + \theta\bar\pi + \bar\theta b + \bar\theta\theta\rho \\ \mathcal{E}_\theta &= l + \theta c + \bar\theta\bar\lambda + \bar\theta\theta\bar c\,, \end{aligned} \tag{80}$$

where we again use the notation in [6]. We define $g_{\mu\nu} = \delta_{\mu\nu}$ and $g_{\theta\bar\theta} = -g_{\bar\theta\theta} = 1$ and define the brackets of the various component fields in (79), (80) so that our superspace variables obey the following Poisson brackets

$$\{\mathcal{E}_\alpha(y_1),\mathcal{A}_\beta(y_2)\} \;=\; -g_{\alpha\beta}\delta(y_1-y_2)\,. \tag{81}$$

As explained in [6], the components of these additional superfields can be identified as the various ghosts that we need to introduce for a fully gauge fixed quantization of the constrained system (77), (78), according to the Batalin–Fradkin algorithm [11]. Consequently we generalize the BRST operator (62) into

$$\begin{aligned}\Omega \to \psi_\mu E^-_\mu + \bar\eta_\mu \pi_\mu + \varphi \mathcal{P} - \bar\eta \pi - c\rho + \bar c \bar\eta \\ = \int \mathrm{d}\bar\theta \, \mathrm{d}\theta \; g^{\alpha\beta} \partial_\theta \mathcal{A}_\alpha \mathcal{E}_\beta \;\sim\; \partial_\theta \;,\end{aligned} \tag{82}$$

which reproduces equation (162) in Appendix B in terms of our variables.

If we now introduce the following conjugation of (82)

$$\begin{aligned}\Omega \;\to\; & e^{-\Phi} \Omega e^{\Phi} \\ = \; & \Omega + \{\Omega, \Phi\} + \frac{1}{2!}\{\{\Omega,\Phi\},\Phi\} \;+\; \cdots\end{aligned} \tag{83}$$

with

$$\Phi \;=\; \int \mathrm{d}\bar\theta \, \mathrm{d}\theta \; \theta (\mathcal{A}^a_\theta \mathcal{D}^{ab}_\mu \mathcal{E}^b_\mu + \frac{1}{2} f^{abc} \mathcal{A}^a_\theta \mathcal{A}^b_\theta \mathcal{E}^c_{\bar\theta}) \;, \tag{84}$$

where the covariant derivative $\mathcal{D}^{ab}_\mu$ is with respect to the Parisi–Sourlas superconnection, we find for the conjugated Ω

$$\Omega \;=\; \int (g^{\alpha\beta} \partial_\theta \mathcal{A}^a_\alpha \mathcal{E}^a_\beta + \mathcal{A}^a_\theta \mathcal{D}^{ab}_\mu \mathcal{E}^b_\mu - \frac{1}{2} f^{abc} \mathcal{A}^a_\theta \mathcal{A}^b_\theta \mathcal{E}^c_{\bar\theta}) \;. \tag{85}$$

Here the first term coincides with the translation operator (82) in the θ-direction, and the two remaining terms have the standard form of a nilpotent BRST operator for the superspace gauge transformation, with

$$\mathcal{G} \;=\; \mathcal{D}_\mu \mathcal{E}_\mu \tag{86}$$

the superspace Gauss law operator and $\mathcal{A}_\theta$, $\mathcal{E}_{\bar\theta}$ viewed as the superspace ghost.

In conclusion, we have identified our effective theory with the Parisi–Sourlas supersymmetric Yang–Mills theory. In particular, we have provided the proper identification of the additional fields $\mathcal{A}_\theta$ and $\mathcal{A}_{\bar\theta}$ in terms of the ghosts that we need for a complete gauge fixing in the topological sector.

We now proceed to show that our Parisi–Sourlas theory confines. For this we fix the SU(N)-gauge symmetry in (72) by adding the gauge fixing term

$$\int \mathrm{d}\bar\theta \, \mathrm{d}\theta \; \frac{1}{2\lambda} (\partial_\alpha \mathcal{A}^\alpha)^2 + \text{ ghosts} \tag{87}$$

as in [5]. Using (73) the Gaussian part that involves gauge fields becomes

$$\frac{1}{2} \int \mathrm{d}\bar\theta \, \mathrm{d}\theta \; \mathcal{A}^\mu \Big(\partial^2_\alpha \delta_{\mu\nu} - (1 - \frac{1}{\lambda}) \partial_\mu \partial_\nu \Big) \mathcal{A}^\nu . \tag{88}$$

If we denote the superspace momentum by $P = (p, \alpha, \bar\alpha)$ and select the superspace analog of the Feynman gauge by setting $\lambda = 1$, we find that the propagator of the superfield (65) is

$$\begin{aligned}\langle \mathcal{A}_\mu \mathcal{A}_\nu \rangle = \frac{\delta_{\mu\nu}}{P^2} \;&=\; \frac{\delta_{\mu\nu}}{p^2 - \kappa \bar\alpha \alpha} \\ &= \delta_{\mu\nu} \left(\frac{1}{p^2} \;+\; \frac{\kappa \bar\alpha \alpha}{p^4} \right) ,\end{aligned} \tag{89}$$

where we used the standard properties of the Grassmann numbers α and $\bar\alpha$. We can further choose to couple $\mathcal{A}_\mu$ to currents of the form

$$(1 + \frac{1}{2} \bar\alpha \alpha) \; J_\mu(p). \tag{90}$$

One then obtains a purely bosonic propagator

$$\delta_{\mu\nu} \left(\frac{1}{p^2} \;+\; \frac{\kappa}{p^4} \right). \tag{91}$$

Indeed, this exhibits the proper infrared $\mathcal{O}(p^{-4})$ behavior that leads to a linear potential between two static sources. Furthermore, this $\mathcal{O}(p^{-4})$ infrared behavior in (89) is unique in the following sense: By demanding locality and gauge invariance, we can generalize the gauge fermion in (69) by expanding it in derivatives of π_μ,

$$\Psi \;=\; \bar{\mathcal{P}}_\mu (D_\nu F_{\mu\nu} + \kappa \pi_\mu + \kappa_1 D_\nu \pi_\nu \pi_\mu + \kappa_2 D^2 \pi_\mu + \cdots) \;. \tag{92}$$

In the infrared limit $p \to 0$ we then conclude that the dominant contribution to the propagator indeed comes from (69).

The infrared behavior of (89) implies that in the topological sector our theory confines. Indeed, previously it has been conjectured [3] that the large distance limit of Yang–Mills vacuum is a medium of randomly distributed color-electric and color-magnetic fields. Quantitatively, this means that in the infrared limit Yang–Mills theory can be approximated by the following set of equations

$$\begin{aligned} D^{ab}_\mu F^b_{\mu\nu} \;&=\; h^a_\nu \\ \langle h^a_\mu(x) h^b_\nu(y) \rangle \;&=\; \delta^{ab}_{\mu\nu} \;,\end{aligned} \tag{93}$$

where the white noise random source h^a_μ describes the random color-electric and color-magnetic vacuum medium; Notice that since gauge invariance implies

$$D_\mu D_\nu F_{\mu\nu} \;=\; 0 \tag{94}$$

for consistency we must interpret the equations (93) on the gauge equivalence classes $\mathcal{A}/\mathcal{G}$.

The equations (93) coincide with (174) in Appendix B, when we take into account the inessential complication (94) and interpret our equations on $\mathcal{A}/\mathcal{G}$ (with a nontrivial moduli). As a consequence, the present construction can be viewed as a first principles derivation of (93) in four-dimensional Yang–Mills theory.

The argument presented in [3] states that as a consequence of the Parisi–Sourlas mechanism, (93) imply an effective dimensional reduction D=4 → D=2 in the infrared limit with the ensuing confinement of color. Indeed, by a direct computation [4] in the full Parisi–Sourlas theory, where the restriction (73) is not imposed, one can show that in the Parisi–Sourlas theory planar Wilson loops obey an area law, with string tension σ determined[1] by κ

$$\sigma \;=\; \frac{1}{4\pi}\kappa N g^2 \;. \tag{95}$$

This is a direct consequence of the Parisi–Sourlas dimensional reduction that relates (72) to the corresponding two-dimensional, ordinary Yang–Mills theory. Hence our results suggest that the qualitative structure developed in [3] is indeed a proper way to describe color confinement in the infrared limit of ordinary Yang–Mills theory.

However, even though we have found indications of an area law and an $\mathcal{O}(p^{-4})$ propagator, these are *not* for the original, physical field A^+_μ. In order to show that we have indeed derived color confinement from first principles, it is necessary to show that the nonperturbative coupling between A^+ and A^- mediates confinement to the physically relevant A^+ sector of our theory. In particular, we need to understand how a mass gap appears in the physical spectrum.

4 Instantons and topological Yang–Mills theory

We shall now proceed to investigate our variant (46) of the (anti-)self-dual approximation to the Yang–Mills partition function, obtained from (42) by replacing the density of the Euler characteristic in (43) by the (anti-) self-dual density (45). We shall show that instead of the Parisi–Sourlas theory, the approximation (46) yields the N=2 supersymmetric Yang–Mills theory. Since the analysis of (46) is more involved than that of (42), in the present section we shall first develop some formalism and derive an appropriate path integral representation of the Euler characteristic (45). For simplicity, we specialize to the anti-self-dual configurations $F^+ = 0$.

4.1 Topological Yang–Mills

The path integral representation of the Euler characteristic

$$\mathcal{X}(\mathcal{A}/\mathcal{G}) \;=\; \sum_{F^+=0} \operatorname{sign}\det||\frac{\delta F^+}{\delta A}|| \tag{96}$$

has been investigated by Atiyah and Jeffrey [12,13]. They showed that (96) coincides with the partition function of topological Yang–Mills theory [7] in the Mathai–Quillen formalism [14]. Their construction is a direct generalization of the familiar result that the Euler characteristic of a compact Riemannian manifold can be represented by the partition function of the de Rham supersymmetric quantum mechanics [13,15]. In the following we shall re-derive their results in a manner that enables us to introduce the N=2 representation of (46).

[1] As a consequence of the restriction (73), the parameter κ becomes in our case merely a gauge fixing parameter, and as such unphysical

The partition function of topological Yang–Mills theory is an example of a cohomological path integral of the form

$$Z_{\rm TYM} \;=\; \int \exp\left(\{\Omega_0, \Psi\}\right) \;, \tag{97}$$

where Ω_0 is a nilpotent BRST operator and Ψ is a gauge fermion. Standard arguments imply that such a path integral describes only the cohomology classes of Ω_0, and it is formally invariant under local variations of Ψ.

In [12] Atiyah and Jeffrey introduced a one-parameter family of Ψ's that interpolates between the Gauss–Bonnet–Chern and the Poincaré–Hopf representatives of the Euler class on $\mathcal{A}/\mathcal{G}$, and for a definite value of the parameter their action reproduces the original action of topological Yang–Mills theory [7]. In the following section we need an explicit representation of their construction. For this we use the notation of Ouvry, Stora and van Baal [16] (except that we denote by $\mathbf{u}$ and $\mathbf{v}$ the standard ghosts for gauge fixing) and introduce a graded symplectic manifold with the following canonical variables

form degree	EVEN (q,p)	ODD (q,p)
0-form	φ , π	$\mathbf{u}$, $\mathbf{v}$
1-form	A, E	ψ, $\mathcal{X}$
2-form	b, c	$\bar\psi$, $\bar{\mathcal{X}}$
4-form	$\bar\varphi$, $\bar\pi$	β, γ

The graded Poisson brackets of these variables are

$$\begin{aligned} \{p^a, q^b\} &= -\,\delta^{ab} \\ \{p^a_\mu, q^b_\nu\} &= -\,\delta^{ab}_{\mu\nu} \\ \{p^a_{\mu\nu}, q^b_{\rho\sigma}\} &= -\,\frac{1}{4}\delta^{ab}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}) \;, \end{aligned} \tag{98}$$

where the 2-form bracket explicitly accounts for the antisymmetry and self-duality of the corresponding variables.

The nilpotent BRST operator Ω_0 that computes the cohomology of the topological Yang–Mills theory is a linear combination [6,16]

$$\Omega_0 \;=\; \Omega_{\rm TOP} \;+\; \Omega_{\rm YM} \;+\; \Omega_{\rm gf} \;. \tag{99}$$

Here

$$\Omega_{\rm YM} \;=\; \mathbf{u}\mathcal{G} \;+\; \frac{1}{2}\mathbf{v}[\mathbf{u},\mathbf{u}] \;+\; h\bar{\mathbf{u}} \tag{100}$$

is the standard nilpotent BRST operator that fixes the SU(N) gauge invariance, with

$$\begin{aligned} \mathcal{G} = D_\mu E_\mu \;&+\; [\varphi,\pi] \;+\; [\bar\varphi,\bar\pi] \;+\; [\beta,\gamma] \\ &+ [\psi_\mu, \mathcal{X}_\mu] \;+\; [\bar\psi_{\mu\nu}, \bar{\mathcal{X}}_{\mu\nu}] \;+\; [b_{\mu\nu}, c_{\mu\nu}] \end{aligned} \tag{101}$$

the Gauss law operator which generates the gauge transformations of the various fields

$$[\mathcal{G}^a, \mathcal{G}^b] \;=\; f^{abc}\mathcal{G}^c \;. \tag{102}$$

The second operator

$$\Omega_{\rm TOP} = \psi_\mu E_\mu \ + \ \varphi(D_\mu \mathcal{X}_\mu + [\bar\psi_{\mu\nu}, c_{\mu\nu}] + \mathbf{v}) + b_{\mu\nu}\bar{\mathcal{X}}_{\mu\nu} \ + \ \gamma[\varphi,\bar\varphi] \ + \ \beta\bar\pi \tag{103}$$

is the BRST operator for the topological symmetry. It is equivariantly nilpotent,

$$\Omega^2_{\rm TOP} \ = \ -2\varphi\mathcal{G} \ . \tag{104}$$

Since this generates a gauge transformation with φ as the gauge parameter, $\Omega_{\rm TOP}$ is nilpotent on the gauge orbit $\mathcal{A}/\mathcal{G}$.

For completeness we have also included in (99) the nilpotent $\Omega_{\rm gf}$ which is necessary for gauge fixing the various symmetries. The structure of $\Omega_{\rm gf}$ in the topological sector has been explained e.g. in [6]. Here we do not need to consider it explicitly, it is sufficient to know that the topological gauge invariances can be fixed by an appropriate gauge condition.

We construct the action of topological Yang–Mills theory by first defining four different gauge fermions Ψ_i

$$\begin{aligned} \Psi_1 &= \bar\psi \wedge b \\ \Psi_2 &= \bar\psi \wedge F^+ \\ \Psi_3 &= \star\bar\varphi \wedge D\star\psi \\ \Psi_4 &= \beta\wedge[\varphi,\star\bar\varphi] \ , \end{aligned} \tag{105}$$

where $\star$ denotes the Hodge duality transformation. We introduce four numerical parameters α_i and define the gauge invariant topological action

$$S_{\rm TOP} \ = \ \sum_{i=1}^4 \ \alpha_i\{\Omega_{\rm TOP},\Psi_i\} \ . \tag{106}$$

By substituting (105) and eliminating the auxiliary field b by a Gaussian integration in (97), we obtain

$$\begin{aligned} -S_{\rm TOP} = {} & \alpha_1\varphi[\bar\psi,\bar\psi] \\ & + \alpha_2\left[-\frac{\alpha_2}{\alpha_1}\frac{1}{4}F^+\wedge F^+ \ + \ D\psi^+\wedge\bar\psi\right] \\ & + \alpha_3\Big(\star\beta\wedge D\star\psi + \star\bar\varphi\wedge[\psi,\star\psi] - \star\bar\varphi D\star D\varphi\Big) \\ & + \alpha_4\Big([\varphi,\star\bar\varphi]^2 + \varphi[\star\beta,\star\beta]\Big)\star 1 \ . \end{aligned} \tag{107}$$

Different values of α_i label different representations of the theory, and by general arguments the ensuing path integral (97) should be independent of these parameters. For example, if we select

$$\alpha_1 \ = \ -\alpha_2 \ = \ \alpha_3 \ = \ 1, \qquad \alpha_4 \ = \ 0 \ , \tag{108}$$

we find the action presented in [16]. On the other hand, the action originally introduced by Witten [7] emerges if we select

$$\alpha_1 \ = \ 4 \ , \ \alpha_2 \ = \ -4 \ , \ \alpha_3 \ = \ 1 \ , \ \alpha_4 \ = \ \frac{1}{2} \ , \tag{109}$$

and use the following identification of variables between the notations in [16] and [7]:

OSvB	Witten
A	A
ψ	$\mathrm{i}\psi$
φ	$\mathrm{i}\phi$
$\bar\psi$	$\frac{1}{4}\mathcal{X}$
$\star\bar\varphi$	$-\frac{\mathrm{i}}{2}\lambda$
$\star\beta$	η

The result is

$$\begin{aligned} -S_W = {} & -\frac{1}{4}F^2_{\mu\nu} \ - \ \frac{1}{4}F_{\mu\nu}\tilde F_{\mu\nu} \ - \ \frac{1}{2}\phi D^2_\mu\lambda \ + \ \mathrm{i}\eta D_\mu\psi_\mu \\ & - \mathrm{i}D_\mu\psi_\nu\mathcal{X}_{\mu\nu} \ + \ \frac{\mathrm{i}}{8}\phi[\mathcal{X}_{\mu\nu},\mathcal{X}_{\mu\nu}] + \ \frac{\mathrm{i}}{2}\lambda[\psi_\mu,\psi_\mu] \\ & + \frac{\mathrm{i}}{2}\phi[\eta,\eta] + \frac{1}{8}[\phi,\lambda]^2 \ . \end{aligned} \tag{110}$$

In [12] Atiyah and Jeffrey showed that the corresponding path integral (97) yields the Euler characteristic (96) of $\mathcal{A}/\mathcal{G}$. For this, we return to the notation of [16] and select

$$\alpha_2 \ = \ -1 \ , \qquad \alpha_4 \ = \ 0 \ , \tag{111}$$

but leave α_1 and α_3 arbitrary. We again eliminate b by a Gaussian integration and find for the path integral (97)

$$\begin{aligned} Z_{\rm TYM} = {} & \int[\mathrm{d}A] \ \cdots \ [\mathrm{d}\lambda][\sqrt{\frac{1}{4\pi\alpha_1}}] \\ & \times \ \exp\Big\{-\int\frac{1}{4\alpha_1}(F^+)^2 + \ D\psi^+\wedge\bar\psi \\ & - \alpha_1\varphi[\bar\psi,\bar\psi] - \ \alpha_3(\star\bar\varphi\wedge[\psi,\star\psi] - \star\bar\varphi D\star D\varphi) \\ & - \alpha_3\star\beta\wedge D\star\psi\Big\} \end{aligned} \tag{112}$$

The parameter α_3 can be eliminated by redefining β and $\bar\varphi$, and we get

$$\begin{aligned} Z_{\rm TYM} = {} & \int[\mathrm{d}A][\mathrm{d}\psi][\mathrm{d}\bar\psi][\mathrm{d}\varphi][\sqrt{\frac{1}{4\pi\alpha_1}}] \\ & \times \ \exp\Big\{-\int\frac{1}{4\alpha_1}(F^+)^2 \ + \ D\psi^+\wedge\bar\psi \\ & - \alpha_1\left(\frac{1}{\star D\star D}\star[\psi,\star\psi]\right)\cdot[\bar\psi,\bar\psi] \ \Big\} \end{aligned} \tag{113}$$

Here

$$R \ = \ \frac{1}{\star D\star D}\star[\psi,\star\psi] \tag{114}$$

can be identified as the curvature 2-form on $\mathcal{A}/\mathcal{G}$ when we restrict ψ to be a horizontal 1-form over $\mathcal{A}/\mathcal{G}$,

$$D\star\psi \ = \ 0 \ , \tag{115}$$

which follows as a δ-function constraint when we integrate over β in (112). Indeed, if we introduce the connection

$$\Gamma \ = \ \frac{1}{\star D\star D}\star D\star\psi \ , \tag{116}$$

and define the exterior derivative by

$$d \;=\; \psi^a_\mu \frac{\delta}{\delta A^a_\mu}\;, \tag{117}$$

we find that the curvature 2-form

$$R \;=\; d\Gamma \;+\; \Gamma\wedge\Gamma \tag{118}$$

coincides with (114) when we restrict to the horizontal bundle (115).

According to general arguments, the path integral (113) is at least formally independent of α_1 and we can interpret it by considering various limits: When $\alpha_1 \to \infty$ we find that (113) reduces to the Gauss–Bonnet–Chern representation of the Euler characteristic on $\mathcal{A}/\mathcal{G}$

$$Z_{\rm TYM} \;=\; \int [\,dA][\,d\psi]{\rm Pf}(R)\;, \tag{119}$$

where R is the curvature 2-form (114). On the other hand, when $\alpha_1 \to 0$ we get

$$Z_{\rm TYM} = \int [\,dA]\delta(F^+)\det||D^+|| \approx \sum_{F^+=0} {\rm sign}\det||\frac{\delta F^+}{\delta A}||\;, \tag{120}$$

which is the Poincaré–Hopf representation (96) of the Euler characteristic on $\mathcal{A}/\mathcal{G}$. As a consequence we have a generalization of the finite dimensional relation between the Gauss–Bonnet–Chern and Poincaré–Hopf theorems, and in particular (114) is indeed the curvature 2-form on $\mathcal{A}/\mathcal{G}$.

We now proceed to discuss how the present results lead to the emergence of the N=2 structure and confinement in the approximation (46).

4.2 The instanton approximation to Yang–Mills

Our instanton approximation replaces the background field representation (39) of the original Yang–Mills partition function (3) by the following summation over anti-self-dual configurations,

$$Z_{\rm YM} \approx \sum_{F^+=0} {\rm sign}\det||\frac{\delta F^+}{\delta A}|| \times \left\{ \int [\,dQ]\exp\{\int -\frac{1}{4}F^2_{\mu\nu}(A+Q)\} \right\}. \tag{121}$$

Evidently the self-dual approximation is similar, and there is no need to consider it explicitly. In order to ensure consistency with cluster decomposition, when necessary we can also introduce the explicit symmetrization (47).

According to (120) the summation over anti-self-dual configurations in (121) can be obtained as a definite limit of the partition function (112) of topological Yang–Mills theory. This implies that (121) coincides with the $\alpha_1 \to 0$ limit of the following more general path integral

$$Z_{\rm YM}(\alpha_1,\alpha_3) = \int [\,dQ][\,dA]\;\cdots\;[\,d\beta] \times\exp\{-S_{\rm YM}(A+Q)\;-\;S_{\rm AJ}(A;\alpha_1,\alpha_3)\}\;, \tag{122}$$

i.e.,

$$\lim_{\alpha_1\to 0} Z_{\rm YM}(\alpha_1,\alpha_3) = Z_{\rm YM}\;. \tag{123}$$

Here $S_{\rm YM}$ is the $A_\mu + Q_\mu$ dependent Yang–Mills background field action that appears in (121) and $S_{\rm AJ}$ is the A_μ dependent Atiyah–Jeffrey representation of the topological Yang–Mills action that appears in (112).

Due to the presence of the Yang–Mills background action, the path integral (122) depends a priori nontrivially on the parameters α_1 and α_3, and (121) and (122) coincide only in the $\alpha_1 \to 0$ limit. However, we now argue that (122) is actually *independent* of α_1 and α_3 so that it coincides with (121) independently of these parameters. More generally, we argue that independently of the parameters α_i the path integral (121) coincides with

$$Z_{\rm YM} \;=\; \int [\,dQ][\,dA]\;\cdots\;[\,d\beta] \times\exp\Big(\int -\frac{1}{4}F^2_{\mu\nu}(A+Q) - \sum_{i=1}^{4}\;\alpha_i\{\Omega_{\rm TOP},\Psi_i\}\Big), \tag{124}$$

where $\Omega_{\rm TOP}$ is the topological BRST operator (103) and Ψ_i are the gauge fermions defined in (105). Here the second term in (125) depends only on the classical field A_μ, and coincides with the action (106) of topological Yang–Mills theory.

This parameter independence of (125) follows immediately when we recall (51) – (54) that $A+Q$ and A can be viewed as two independent gauge fields A^+ and A^-. The action in (125) then separates into two independent contributions, the first term depends only on the gauge field A^+_μ in (54) and the second term depends only on the gauge field A^-_μ in (54)

$$S_{\rm YM}(A^+) + S_{\rm TOP}(A^-) \;=\; \frac{1}{4}F^2_{\mu\nu}(A^+) \;+\; \sum_{i=1}^{4}\alpha_i\{\Omega^-_{\rm TOP},\Psi_i\}(A^-)\;. \tag{125}$$

In particular, the BRST operator $\Omega^-_{\rm TOP}$ acts only on the fields A^- and E^-

$$\Omega^-_{\rm TOP} = \psi_\mu E^-_\mu \;+\; \varphi(D^-_\mu \mathcal{X}_\mu + [\bar\psi_{\mu\nu},c_{\mu\nu}] + \rho) \;+\; b_{\mu\nu}\bar{\mathcal{X}}_{\mu\nu} \;+\; \gamma[\varphi,\bar\varphi] \;+\; \beta\bar\pi\;. \tag{126}$$

For $\tau = 0$ (126) reduces to (99) except for the (irrelevant) shift $E_\mu \to E_\mu - P_\mu$. Since τ only parametrizes a canonical transformation this establishes the asserted α_i-independence of the path integral (125). In particular, by selecting $\alpha_2 = -1$, $\alpha_4 = 0$ and taking the $\alpha_1 \to 0$ limit we obtain our anti-self-dual approximation (121).

As in Sect. 3, the present construction implies that instead of the original SU(N) gauge symmetry we now have two *independent* SU(N) gauge symmetries acting on the fields $A^{\pm}$ respectively. In order to properly eliminate the $\mathrm{SU(N)}_+ \times \mathrm{SU(N)}_-$ gauge invariance, we need a BRST operator for both SU(N): In addition of the BRST operator (100) which eliminates the $\mathrm{SU(N)}_-$ gauge invariance

$$\Omega^-_{\mathrm{YM}} \;=\; \mathbf{u}\mathcal{G}^- \;+\; \frac{1}{2}\mathbf{v}[\mathbf{u},\mathbf{u}] \;+\; h\bar{\mathbf{u}}\,, \tag{127}$$

where the Gauss law operator $\mathcal{G}^-$ is now

$$\mathcal{G}^- \;=\; D^-_\mu E^-_\mu + [\varphi,\pi] \;+\; [\bar\varphi,\bar\pi] \;+\; [\beta,\gamma] \\ +\; [\psi_\mu,\mathcal{X}_\mu] \;+\; [\bar\psi_{\mu\nu},\bar{\mathcal{X}}_{\mu\nu}] \;+\; [b_{\mu\nu},c_{\mu\nu}]\,, \tag{128}$$

we also introduce the following nilpotent BRST operator for the $\mathrm{SU(N)}_+$ gauge group

$$\Omega^+_{\mathrm{YM}} \;=\; \mathbf{c}D^+_\mu E^+_\mu \;+\; \frac{1}{2}\mathbf{b}[\mathbf{c},\mathbf{c}] \;+\; t\bar{\mathbf{c}}\,. \tag{129}$$

We then add the corresponding gauge fixing terms to our action,

$$S_{\mathrm{YM}}(A^+) \;+\; S_{\mathrm{TOP}}(A^-) = \\ \int \frac{1}{4}F^2_{\mu\nu}(A^+) \;+\; \sum_{i=1}^{4}\alpha_i\{\Omega^-_{\mathrm{TOP}},\Psi_i\} \;+\; \{\Omega^+_{\mathrm{YM}},\Psi^+\} \\ +\;\{\Omega^-_{\mathrm{YM}},\Psi^-\}\,. \tag{130}$$

Here Ω^-_{TOP} and Ω^-_{YM} depend only on A^-, E^- while Ω^+_{YM} depends only on A^+, E^+. Hence Ω^+_{YM} anticommutes with both Ω_{TOP} and Ω^-_{YM} and the ensuing path integral is invariant under local variations of the gauge fermions $\Psi_i(A^-)$, $\Psi^{\pm}(A^{\pm})$.

In summary, we have established that our instanton approximation (46) involves the topological Yang–Mills theory. Since we expect our instanton approximation to adequately describe the Yang–Mills partition function and since the Parisi–Sourlas theory confines, this suggests that topological Yang–Mills theory also confines. In the next section we proceed to show that this is indeed the case: We shall show that our topological Yang–Mills theory yields the minimal N=2 supersymmetric Yang–Mills theory at points of its moduli space where confinement occurs.

5 N=2 supersymmetry and color confinement

We shall now proceed to relate the previous construction to the minimal N=2 supersymmetric Yang–Mills theory. Indeed, topological Yang–Mills theory is a twisted version of the N=2 theory, obtained by reinterpreting the action of Lorentz transformations [7]. However, for the present purposes we need a refinement of this connection: In the minimal N=2 supersymmetric theory confinement is only known to occur at special points in the N=2 quantum moduli space that correspond to massless dyons. Since we wish to argue that there is a connection between confinement in the N=2 theory and color confinement in ordinary Yang–Mills theory, we need to establish that (un)twisting actually takes us to such special points of the N=2 moduli space.

The (Minkowski space) action of the minimal SU(N) invariant N=2 supersymmetric Yang–Mills theory is

$$-S_{\mathrm{N=2}} \;=\; -\,\frac{1}{4}F^2_{\mu\nu} \;-\; D_\mu B D_\mu \bar B \;-\; \mathrm{i}\bar\lambda_i\bar\sigma_\mu D_\mu\lambda^i \\ -\;\frac{1}{\sqrt2}B[\bar\lambda_i,\bar\lambda^i] \;+\; \frac{1}{\sqrt2}\bar B[\lambda_i,\lambda^i] + \frac{1}{2}[B,\bar B]^2\,. \tag{131}$$

By comparing the actions (110) and (131) term-by-term, we observe an obvious similarity. Indeed, since the Lorentz algebra SO(3,1) is related to SO(4) $\sim$ SU(2) $\times$ SU(2) and since the action (131) has an internal SU(2) symmetry, we can re-define the action of Lorentz transformations [7]. Specializing to SU(2), at the level of fields this means that we look for an invertible change of variables between (110) and (131)

$$(B,\bar B,\lambda_i,\bar\lambda_i,A_\mu) \;\to (\phi,\lambda,\eta,\psi_\mu,\mathcal{X}_{\mu\nu},A_\mu)\,, \tag{132}$$

which maps (131) into (110) and vice versa. By direct comparison of the actions we conclude that this change of variables is defined by

$$\begin{aligned} B &= -\,\mathrm{i}\sqrt2\phi \\ \bar B &= -\,\frac{\mathrm{i}}{\sqrt8}\lambda \\ \lambda_{\alpha i} &= -\,\sigma^\mu_{\alpha i}\psi^\mu \\ \bar\lambda_{\dot\alpha i} &= -\,\frac{1}{2}\epsilon_{\dot\alpha i}\eta + \frac{1}{4}\bar\sigma^{\mu\nu}_{\dot\alpha i}\mathcal{X}_{\mu\nu}\,, \end{aligned} \tag{133}$$

and the inverse transformation is

$$\begin{aligned} \phi &= \frac{\mathrm{i}}{\sqrt2}B \\ \lambda &= \mathrm{i}\sqrt8\bar B \\ \eta &= \bar\lambda_i{}^i \\ \psi_\mu &= \frac{1}{2}\sigma^{\alpha i}_\mu\lambda_{\alpha i} \\ \mathcal{X}_{\mu\nu} &= 2\bar\sigma^{\dot\alpha i}_{\mu\nu}\bar\lambda_{\dot\alpha i}\,. \end{aligned} \tag{134}$$

Indeed, if we substitute (133) in the N=2 action (131) (modulo analytic continuation to the Euclidean space and the topological $F\tilde F$ term) we find

$$-S = -\frac{1}{4}F^2_{\mu\nu} \;-\; \frac{1}{2}\phi D^2_\mu\lambda \;+\; \mathrm{i}\eta D_\mu\psi_\mu \;+\; \mathrm{i}\mathcal{X}_{\mu\nu}D_\mu\psi_\nu \\ +\;\frac{\mathrm{i}}{8}\phi[\mathcal{X}_{\mu\nu},\mathcal{X}_{\mu\nu}] \;+\; \frac{\mathrm{i}}{2}\lambda[\psi_\mu,\psi_\mu] \\ +\;\frac{\mathrm{i}}{2}\phi[\eta,\eta] \;+\; \frac{1}{8}[\phi,\lambda]^2\,, \tag{135}$$

which is the action (110) of the topological Yang–Mills theory. Since this change of variables has a trivial Jacobian in the path integral, we conclude that the ensuing partition functions coincide, only the interpretation of these two theories is different.

Similarly we can introduce a change of variables which relates the N=2 theory to the self-dual topological Yang–Mills theory instead of the anti-self-dual one.

Next we consider the pertinent supersymmetry algebras. In the N=2 theory we have

$$\begin{aligned} \{Q_\alpha{}^i, \bar{Q}_{\dot\beta j}\} &= 2\delta^i{}_j \sigma^\mu_{\alpha\dot\beta}\, P_\mu \\ \{Q_\alpha{}^i, Q_\beta{}^j\} &= \epsilon_{\alpha\beta}\epsilon^{ij} Z \\ \{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta j}\} &= -\epsilon_{\dot\alpha\dot\beta}\epsilon_{ij} Z^\star \, , \end{aligned} \tag{136}$$

where

$$Z \sim a n_e + a_D n_m \tag{137}$$

is the central charge, and we have used the notation in [1]. Here a is a complex coordinate of the N=2 moduli space. It labels different Higgs vacua and for $a \neq 0$ the SU(2) gauge symmetry becomes spontaneously broken to U(1). The parameter a_D is the corresponding dual variable, and (n_e, n_m) are the electric and magnetic charges. According to [1] the zeroes of (137) are of special importance: These are the points where dyons become massless and may condense, implying confinement according to the picture developed in [2].

We shall now argue that it is exactly these $Z = 0$ points of the moduli space that appear in our construction. For this we proceed to represent the N=2 supersymmetry algebra (136) in terms of the topological variables:

By applying (133) in (136) we find that the BRST operator of the topological Yang–Mills theory is related to the N=2 supersymmetry generators by

$$\Omega = \epsilon^{\dot\alpha i} \bar{Q}_{\dot\alpha i} \, . \tag{138}$$

Furthermore, since

$$\{Q_1{}^2, \bar{Q}_{\dot 1 2}\} = -2(H - P_3) \tag{139}$$

and

$$\{Q_2{}^1, \bar{Q}_{\dot 2 1}\} = -2(H + P_3) \, , \tag{140}$$

where H is the Hamiltonian of (131), combining (139) and (140) we get

$$H = -\frac{1}{4}\{\bar{Q}_{\dot 1 2} - \bar{Q}_{\dot 2 1}, Q_1{}^2 - Q_2{}^1\} = \{\Omega, \Psi\} \, . \tag{141}$$

This implies that the gauge fermion Ψ that yields the action (135) of topological Yang–Mills theory is

$$\Psi = \frac{1}{4}(Q_2{}^1 - Q_1{}^2) \, . \tag{142}$$

If we further define the following two operators

$$\begin{aligned} Q^\mu &= \sigma^\mu_{\alpha i} \epsilon^{\alpha\beta} Q_\beta{}^i \\ D^{\mu\nu} &= \bar\sigma^{\mu\nu\dot\alpha}_{\dot\beta} \epsilon^{\dot\beta i} \bar{Q}_{\dot\alpha i} \, , \end{aligned} \tag{143}$$

and introduce the anti-self-dual projection operator

$$\mathcal{P}_-{}^{\mu\nu\rho\sigma} = \frac{1}{4}\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} - i\epsilon^{\mu\nu\rho\sigma}\right) \, , \tag{144}$$

we find that in terms of the twisted variables the supersymmetry algebra (136) becomes

$$\begin{aligned} \{\Omega, \Omega\} &= -2Z^\star \\ \{\Omega, Q^\mu\} &= 4P^\mu \\ \{\Omega, D^{\mu\nu}\} &= 0 \\ \{Q^\mu, Q^\nu\} &= 2\eta^{\mu\nu} Z \\ \{Q^\mu, D^{\rho\sigma}\} &= 8P_\nu \mathcal{P}_-{}^{\rho\sigma\mu\nu} \\ \{D^{\mu\nu}, D^{\rho\sigma}\} &= -2Z^\star \mathcal{P}_-{}^{\mu\nu\rho\sigma} \, . \end{aligned} \tag{145}$$

In particular, we find that if the central charge

$$Z \sim a n_e + a_D n_m \tag{146}$$

is nonvanishing, the BRST operator is not nilpotent. However, since our starting point is topological Yang–Mills theory where Ω must be nilpotent and since (in each instanton sector) the N=2 theory differs from the original topological theory only by an invertible change of variables, we conclude that in the present case we must necessarily have

$$Z = 0 \, . \tag{147}$$

This means that we must restrict our twisted N=2 theory to a point in its moduli space where the central charge Z vanishes, i.e.

$$a n_e = -a_D n_m \, . \tag{148}$$

For color confinement the Hamiltonian in the ordinary Yang–Mills theory must exhibit a mass gap. This ensures that the physical degrees of freedom are not massless gluons, but massive composites such as a glueball. In the N=2 theory a nonvanishing a introduces a mass scale, and generates a mass for the non-abelian components of the gauge field by Higgs mechanism. Consequently it is natural to consider such points in the N=2 quantum moduli space where $a \neq 0$.

The condition (148) implies that massless dyons are present in the spectrum of our N=2 theory. It is natural to expect that these massless dyons condense, and this leads to confinement [1]. Indeed, this is consistent with our construction: We have found that in our instanton approximation the Parisi–Sourlas theory becomes replaced by topological Yang–Mills theory. Since the Parisi–Sourlas theory confines, it is natural to expect that the topological theory also confines. Since (un)twisting of the topological theory yields the minimal N=2 theory at points of its quantum moduli space where massless dyons emerge, our result is fully consistent with the condensation of massless dyons [1].

6 Comparison

In the previous sections we have studied the Yang–Mills partition function in two different approaches. We have first investigated the summation over all possible background field configurations which yields an effective field theory that involves the Parisi–Sourlas supersymmetric

Yang–Mills theory. We have then investigated an instanton approximation which yields a description based on the topological and the N=2 supersymmetric Yang–Mills theories.

Since the topological Yang–Mills theory is related to the N=2 supersymmetric Yang–Mills theory by the invertible change of variables (133), the Parisi–Sourlas Yang–Mills theory must also contain the N=2 theory, *both* in its self-dual and anti-self-dual subsectors. This means in particular that the confinement mechanism which has been identified in the N=2 theory [1] must admit an interpretation in terms of the D=4 → D=2 Parisi–Sourlas dimensional reduction. In particular, there should be a direct relation between the picture of confinement by dyon condensation developed in [1,2] and the picture of confinement by randomly fluctuating color-electric and color-magnetic fields developed in [3]. Furthermore, the derivation [4] that planar Wilson loops in the Parisi–Sourlas theory obey an area law should also be directly applicable to the corresponding Wilson loops in the N=2 approach, modulo a change of variables that originates from the different choice of gauge fermions Ψ.

Finally, we need to consider the off-shell phase factors that appear in (41): Since these phase factors depend only on the topological connection A^-, they do not break the $SU_+(N) \times SU_-(N)$ gauge symmetries. However, they do explicitly break the BRST supersymmetries, and in particular the Ψ independence in the topological sectors, but in a controllable fashion. Since the BRST transformation is a change of variables of the form

$$\phi^a \ \to \ \phi^a \ + \ \delta\Psi\{\Omega, \phi^a\} \ , \tag{149}$$

we conclude that when we vary the gauge fermion $\Psi \to \Psi + \delta\Psi$ the phase factors do not remain intact but suffer a nontrivial change of variables. Besides the terms that we have discussed in the previous sections, we should then add the phase factors which have been subjected to the proper changes of variables. These are additional non-local terms that should be included to our action. However, since these terms are irrelevant in the infrared limit, we need to include them only when we consider higher order corrections.

Conclusion

We have applied background field formalism in the path integral approach to investigate confinement in the ordinary Yang–Mills theory. By comparing the summation over all background fields to the summation over self-dual fields, we have found an intimate relationship between the Parisi–Sourlas and N=2 approaches to confinement. In particular, our results suggest that the N=2 approach to confinement can also be interpreted in terms of the Parisi–Sourlas dimensional reduction. Furthermore, the qualitative picture of the N=2 approach in terms of massless dyon condensation should coincide with the Parisi–Sourlas picture where the Yang–Mills vacuum is viewed as a medium of randomly fluctuating color-electric and color-magnetic fields.

Acknowledgements. J.K. thanks O. Tirkkonen and A.N. thanks L. Faddeev, A. Polyakov, G. Semenoff, N. Weiss and A. Zhitnitsky for discussions. We both thank G. Semenoff and the Department of Physics at University of British Columbia for hospitality during this work.

Appendix A Classical Morse theory

In Sect. 2 we have motivated our instanton approximation (46) using an analogy with Morse theory. For completeness, we here review the relevant aspects of classical Morse theory [10].

In classical Morse theory we are interested in the critical point structure of a function $H(x)$ called the Morse function, defined on a compact manifold. For simplicity, we assume that the critical points $dH = 0$ are isolated and nondegenerate. A sum such as (40)

$$\sum_{dH=0} 1 \tag{150}$$

counts the number of critical points, and is bounded from below by the sum of Betti numbers B_n of the underlying manifold

$$\sum_{dH=0} 1 \ \geq \ \sum_n B_n \ . \tag{151}$$

On the other hand, a sum such as the r.h.s. of (19) is independent of the Morse function H and according to the Poincaré–Hopf theorem [10] it coincides with the Euler characteristic $\mathcal{X}$ of the manifold

$$\sum_{dH=0} \text{sign}\det || \frac{\partial^2 H}{\partial x_a \partial x_b} || \ = \ \sum_n (-)^n B_n \ \equiv \ \mathcal{X} \ . \tag{152}$$

If H is a perfect Morse function these two quantities coincide, but for a general Morse function they are different since in general the fluctuation matrix $\partial_{ab} H$ admits an odd number of zero modes for some of the critical points x_a.

For a compact finite dimensional Riemannian manifold the Euler characteristic (152) coincides with the partition function of the de Rham supersymmetric quantum mechanics [13,15]. (See also Appendix B.) This partition function can be evaluated exactly, e.g. by localizing the corresponding path integral to the Euler class of the manifold. In this way we obtain the standard relation between the Poincaré–Hopf and Gauss–Bonnet–Chern theorems [13].

On the other hand, the summation that appears in (46) is an infinite dimensional generalization of a sum of the form

$$\sum_{dH=0} \text{sign}\det || \frac{\partial^2 H}{\partial x_a \partial x_b} || \ \exp\{-T\mathcal{H}\} \ , \tag{153}$$

where $\mathcal{H}$ corresponds to the Q-integral in (46). When H and $\mathcal{H}$ coincide, we obtain a quantity that appears in an equivariant version of the Poincaré–Hopf theorem [17].

There is also an equivariant version of the Gauss–Bonnet–Chern theorem, and as in conventional Morse theory one can derive a relation between these two theorems using an equivariant version of the de Rham supersymmetric quantum mechanics [17]. The pertinent path integral is intimately related to a standard Hamiltonian path integral, with the Morse function H interpreted as the Hamiltonian function. Indeed, this interrelationship between equivariant Morse theory and standard Hamiltonian path integrals can be utilized to evaluate certain Hamiltonian partition functions exactly, using localization techniques [18]. This leads to the Duistermaat–Heckman integration [14] formula and its quantum mechanical generalizations [17,18].

Consequently we expect that (46) provides a good approximation of the original partition function (39).

Appendix B Quantum Morse theory and Parisi–Sourlas supersymmetry

In the present article we have investigated the infrared limit of the Yang–Mills partition function. In our construction we have employed quantities such as the Poincaré–Hopf representation of the Euler characteristic

$$\begin{aligned}\mathcal{X}(\mathcal{A}/\mathcal{G}) &= \sum_{DF=0} \mathrm{sign}\det ||\frac{\delta DF}{\delta A}|| \\ &= \sum_{F^\pm} \mathrm{sign}\det ||\frac{\delta F^\pm}{\delta A}|| \end{aligned} \tag{154}$$

on the gauge equivalence classes $\mathcal{A}/\mathcal{G}$. In this appendix we shall relate such Morse theoretic quantities to the Parisi–Sourlas formalism. For this we consider a generic D-dimensional quantum field theory defined by an action $S(\phi^a)$, where $\{\phi^a\}$ are fields that take values on some configuration space which in the case of the Yang–Mills theory is $\mathcal{A}/\mathcal{G}$. For simplicity, we shall assume that $S(\phi^a)$ has the same formal properties as a nondegenerate Morse function.

According to the Poincaré–Hopf theorem the Euler characteristic of the $\{\phi^a\}$ field space can be represented as

$$\begin{aligned}\mathcal{X}(\phi) &= \sum_{\delta S=0} \mathrm{sign}\det ||\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}|| \\ &= \int [\mathrm{d}\phi]\delta\left(\frac{\delta S}{\delta\phi^a}\right)\det ||\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}|| \\ &= \int [\mathrm{d}\phi][\mathrm{d}\pi][\mathrm{d}\psi][\mathrm{d}\bar{\mathcal{P}}] \\ &\quad \times \exp\{\mathrm{i}\int \pi^a\frac{\delta S}{\delta\phi^a} + \psi^a\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}\bar{\mathcal{P}}^b\} \, . \end{aligned} \tag{155}$$

A priori this is independent of the Morse function $S(\phi)$. Here we have introduced one commuting (π^a) and two anticommuting (ψ^a, $\bar{\mathcal{P}}^a$) auxiliary variables, to exponentiate the δ-function and the determinant respectively. The action in (155)

$$S_{\mathrm{eff}} \;=\; \int \pi^a\frac{\delta S}{\delta\phi^a} \;+\; \psi^a\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}\bar{\mathcal{P}}^b \tag{156}$$

has the following nilpotent supersymmetry

$$\begin{aligned}\Omega\phi^a &= \psi^a \\ \Omega\psi^a &= 0 \\ \Omega\bar{\mathcal{P}}^a &= \pi^a \\ \Omega\pi^a &= 0 \, , \end{aligned} \tag{157}$$

so that we can represent Ω by

$$\Omega \;=\; \psi^a\frac{\delta}{\delta\phi^a} \;+\; \pi^a\frac{\delta}{\delta\bar{\mathcal{P}}^a} \, , \tag{158}$$

and clearly

$$\Omega^2 \;=\; 0 \, . \tag{159}$$

We identify (157) as the Parisi–Sourlas supersymmetry [5], when realized on a scalar superfield in the Parisi–Sourlas superspace. In addition to the space coordinates x, this superspace has two anticommuting coordinates θ and $\bar\theta$

$$\theta^2 \;=\; \bar\theta^2 \;=\; \theta\bar\theta + \bar\theta\theta \;=\; 0 \, . \tag{160}$$

The scalar superfield is

$$\Phi^a(x,\theta,\bar\theta) \;=\; \phi^a + \theta\psi^a - \bar\theta\bar{\mathcal{P}}^a + \theta\bar\theta\pi^a \, , \tag{161}$$

and the supersymmetry (157) can be identified with the translation in the θ direction of the superspace,

$$\Omega \;=\; \int \partial_\theta\Phi^a\frac{\delta}{\delta\Phi^a} \;\sim\; \partial_\theta \, . \tag{162}$$

Using (161) we can write the action in (155) as

$$\begin{aligned}\int \mathrm{d}x\,\left(\pi^a\frac{\delta S}{\delta\phi^a} \;+\; \psi^a\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}\bar{\mathcal{P}}^b\right) &= \\ \int \mathrm{d}x\,\mathrm{d}\bar\theta\,\mathrm{d}\theta\; S(\Phi) \, . \end{aligned} \tag{163}$$

Hence, as a function of the superfield Φ^a the action (156) has the same functional form as our original action $S(\phi)$. Furthermore, if we introduce the following functional

$$\Psi \;=\; \bar{\mathcal{P}}^a\frac{\delta S}{\delta\phi^a} \, , \tag{164}$$

we find that (156) is closed under the supersymmetry operator (158)

$$\int \pi^a\frac{\delta S}{\delta\phi^a} + \psi^a\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}\bar{\mathcal{P}}^b \;=\; \int \Omega\Psi. \tag{165}$$

Consequently, the path integral (155) is of the standard cohomological form (97)

$$\mathcal{X}(\phi) \;=\; \int [\mathrm{d}\phi][\mathrm{d}\pi][\mathrm{d}\bar{\mathcal{P}}][\mathrm{d}\psi]\exp\{\mathrm{i}\int \Omega\Psi\} \, , \tag{166}$$

and in particular it is invariant under local variations of the gauge fermion Ψ. Notice that this ensures that the Euler characteristic (155) is indeed independent of the local details of the "Morse functional" $S(\phi)$.

We shall now apply the Ψ-invariance of (166) to generalize (164) to

$$\Psi \;=\; \bar{\mathcal{P}}^a(\frac{\delta S}{\delta\phi^a} + \frac{\kappa}{2}\pi^a) \;, \tag{167}$$

where κ is a parameter. For the action this yields

$$\int \Omega\Psi \;=\; \int \pi^a\frac{\delta S}{\delta\phi^a} \;+\; \psi^a\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}\bar{\mathcal{P}}^b \;-\; \frac{\kappa}{2}\pi^2 \;, \tag{168}$$

and in terms of the superfield Φ we get

$$\int \Omega\Psi \;=\; \int \mathrm{d}x\,\mathrm{d}\bar\theta\,\mathrm{d}\theta\left\{S(\Phi) - \frac{\kappa}{2}\Phi^a\partial_{\bar\theta}\partial_\theta\Phi^a\right\} \;. \tag{169}$$

If the original action $S(\phi)$ has the standard functional form

$$S(\phi) \;=\; \frac{1}{2}\phi^a(-\Box)\phi^a \;+\; V(\phi) \;, \tag{170}$$

we then conclude that the superspace action can be represented in the corresponding superspace form

$$S(\Phi) \;=\; \frac{1}{2}\Phi^a(-\Box - \kappa\partial_{\bar\theta}\partial_\theta)\Phi^a \;+\; V(\Phi) \;. \tag{171}$$

This is the standard Parisi–Sourlas action for a scalar field theory that has been extensively investigated in [5]. In particular, it has been established that in perturbation theory the superspace quantum field theory determined by (171) coincides diagram-by-diagram with the quantum field theory of (170), but in two less space-time dimensions. This D $\to$ D-2 dimensional reduction is a consequence of the negative dimensionality of the anticommuting coordinates [5]. The dimensional transformation from the D-dimensional coupling constants etc. to their (D-2)-dimensional counterparts is determined by the parameter κ which has the dimensions

$$[\kappa] \;\propto\; m^2 \;, \tag{172}$$

when we define the anticommuting variables θ and $\bar\theta$ to be dimensionless. The overall numerical scale of κ is undetermined, and can be changed by redefining the normalization of the θ, $\bar\theta$ integral.

As explained in [5], the superspace quantum theory can also be interpreted in terms of a stochastic differential equation. For this we introduce an additional variable h and write the path integral (166), (167) in the following equivalent form

$$\begin{aligned}\mathcal{X}(\phi) &= \int[\frac{1}{2\sqrt{\kappa}}\,\mathrm{d}h][\,\mathrm{d}\phi]\delta(\frac{\delta S}{\delta\phi^a} - h^a)\det||\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}|| \\ &\quad\times\exp\{-\int\frac{1}{4\kappa}h^2\} \\ &= \int[\frac{1}{2\sqrt{\kappa}}\,\mathrm{d}h]\exp\{-\int\frac{1}{4\kappa}h^2\} \\ &\quad\times\sum_{\frac{\delta S}{\delta\phi^a}=h^a}\mathrm{sign}\det||\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}|| \;.\end{aligned} \tag{173}$$

This has the interpretation of averaging classical solutions to the stochastic differential equation

$$\frac{\delta S}{\delta\phi^a} \;=\; -\Box\phi^a \;+\; \partial_a V(\phi) \;=\; h^a \tag{174}$$

over the external Gaussian random source h. Notice that as a consequence of the Ψ invariance the integral (173) is actually independent of κ, and if we take the $\kappa \to 0$ limit and recall the Gaussian definition of a δ-function we find that (173) reduces to (155). This is fully consistent with the fact that the Euler characteristic is independent of the Morse functional. Indeed, from the Morse theory point of view

$$S_h(\phi) \;=\; S(\phi) \;+\; h^a\phi^a \tag{175}$$

is simply another (nondegenerate) Morse functional. (Here we assume that the $h^a\phi^a$ term is a small perturbation in the sense described in [19].)

Finally, we conclude this section by deriving the Gauss–Bonnet–Chern theorem that represents $\mathcal{X}(\phi)$ in terms of the curvature 2-form on the configuration space $\{\phi^a\}$. (Here we assume that ϕ^a has a nontrivial topology; If the configuration space is a flat Euclidean manifold, see [19]). For this we introduce a canonical transformation, determined by conjugating Ω

$$\Omega \;\to\; \mathrm{e}^{-U}\,\Omega\,\mathrm{e}^{U} \;. \tag{176}$$

We select

$$U \;=\; -\Gamma^a_{bc}\psi^c\bar{\mathcal{P}}_a\lambda^b \;, \tag{177}$$

where we have introduced the conjugate variable

$$\{\pi^a, \lambda^b\} \;=\; -\delta^{ab} \;, \tag{178}$$

and $\Gamma^a_{bc}(\phi)$ are components of a connection on the configuration space $\{\phi^a\}$. For the conjugated Ω we find the following transformation laws,

$$\begin{aligned}\Omega\phi^a &= \psi^a \\ \Omega\psi^a &= 0 \\ \Omega\bar{\mathcal{P}}_a &= \pi_a \;+\; \Gamma^c_{ab}\psi^b\bar{\mathcal{P}}_c \\ \Omega\pi_a &= \Gamma^c_{ab}\pi_c\psi^b \;-\; \frac{1}{2}R^c{}_{adb}\psi^b\psi^d\bar{\mathcal{P}}_c \;,\end{aligned} \tag{179}$$

which we identify as the familiar transformation law of the standard (N=1) de Rham supersymmetric quantum mechanics [17]. Indeed, if we assume that the connection Γ^a_{bc} is metric and use the metric tensor $g_{ab}(\phi)$ to define

$$\Psi \;=\; g^{ab}\pi_a\bar{\mathcal{P}}_b \;, \tag{180}$$

we immediately find that the corresponding path integral (166) evaluates to

$$\mathcal{X}(\phi) \;=\; \int[\,\mathrm{d}\phi][\,\mathrm{d}\psi]\;\mathrm{Pf}[\frac{1}{2}R^a{}_{bcd}\psi^c\psi^d] \;. \tag{181}$$

This is the formal (functional) Euler characteristic of the configuration space $\{\phi^a\}$, and establishes that the Gauss–Bonnet–Chern representation of the Euler characteristic indeed coincides with the Poincaré–Hopf representation (155), (173).

References

1. N. Seiberg, E. Witten: Nucl. Phys. **B431** (1994) 484; ibid. **B426** (94) 19
2. G. 't Hooft: Nucl. Phys. **B190** (1981) 455; ibid. **B153** (1979) 141; A. Polyakov: Nucl. Phys. **B120** (1977) 429
3. H.B. Nielsen, P. Olesen: preprint NBI-HE-79-45 (unpublished); P. Olesen: Nucl. Phys. **B200** (1981) 82; P. Olesen: Phys. Scripta **23** (1981) 1000; G. Parisi: in *High Energy Physics - 1980: Proceedings* (L. Durand and L.G. Pondrom, eds.) (AIP 1981)
4. A. Kupiainen, A.J. Niemi: Phys. Lett. **120B** (1983) 399; T.I. Belova, Yu.M. Makeenko, M.I. Polikarpov, A.I. Veselov: Nucl. Phys. **B230** (1984) 473
5. G. Parisi, N. Sourlas: Phys. Rev. Lett. **43** (1979) 744; B. McClain, A.J. Niemi, C. Taylor, L.C.R. Wijewardhana: Nucl. Phys. **B217** (1983) 430; Ann. Phys. **140** (1982) 232
6. J.M. Maillet, A.J. Niemi: Phys. Lett. **B223** (1989) 195; A. Hietamäki, A.J. Niemi, O. Tirkkonen: Phys. Lett. **B238** (1990) 291
7. E. Witten: Comm. Math. Phys. **117** (1988) 353
8. J. Honerkamp: Nucl. Phys. **B48** (1972) 269
9. L.F. Abbott: Nucl. Phys. **B185** (1981) 189
10. J. Milnor: *Morse Theory*, Princeton University Press, 1973
11. I.A. Batalin, E.S. Fradkin: Phys. Lett. **122B** (1983) 157; M. Henneaux: Phys. Rept. **128** (1986) 1
12. M.F. Atiyah, L. Jeffrey: J. Geom. Phys. **7** (1990) 120
13. D. Birmingham, M. Blau, M. Rakowski, G. Thompson: Phys. Rept. **209** (1991) 129
14. N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Springer Verlag, Berlin, 1991
15. E. Witten: Nucl. Phys. **B202** (1982) 253
16. S. Ouvry, R. Stora, P. van Baal: Phys. Lett. **B220** (1989) 1590
17. A.J. Niemi, K. Palo: `hep-th/9406068`
18. M. Blau, E. Keski-Vakkuri, A.J. Niemi: Phys. Lett. **B246** (1990) 92; A.J. Niemi, O. Tirkkonen: Phys. Lett. **B293** (1992) 339
19. A.J. Niemi: Phys. Lett. **B355** (1995) 501
20. C.H. Llewellyn Smith: Phys. Lett. **46B** (1973) 233; J.M. Cornwall, D.N. Levin, G. Tiktopoulos: Phys. Rev. Lett. **30** (1973) 1268; Phys. Rev. **D10** (1974) 1145